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Group characters, permutation actions and sharpness

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Abstract

We extend the work which has appeared in papers on sharp characters and originated with Blichfeldt and Maillet to the Burnside ring of a finite group G. We show that the polynomial whose zeros are the set of marks of non-identity subgroups on a faithful G-set X evaluated at X is an integral multiple of the regular G-set, and deduce a result about the size of a base of X. Further consequences for ordinary group characters are obtained by re-examining Blichfeldt's work and we provide alternative definitions of sharpness. Conjectures are given related to the set of values of a permutation character, and it is proved that for a faithful transitive G-set X certain polynomials (in the Burnside ring) evaluated at X necessarily give G-sets.

MSC: primary 20C15; secondary 20C20

1. Introduction

In the following, if $L = \{l_1, l_2, \ldots, l_r\}$ is a set of complex numbers the polynomial $f_L(x)$ will be taken to be $\prod_{j=1}^{r}(x - l_j)$. A result of Blichfeldt ([2]) implies that if L is the set of distinct values taken on by a permutation character of degree n of the group G on the non-identity elements, then $f_L(n) = m|G|$ where m is an integer. This result was rediscovered and set in a modern context by Cameron and Kiyota in [3] in response to a conjecture of Deza. Cameron and Kiyota showed that Blichfeldt's result follows from the fact that if χ is any generalised character of G and L is the set of distinct values taken on by χ on non-identity elements then

$$f_L(\chi) = m\rho \tag{1}$$

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where ρ is the regular character, and Alvis pointed out that this leads to a generalisation of the well-known Burnside result that if χ is a faithful character of a group with r distinct values then each irreducible character is a constituent of at least one of $\chi, \chi^2, \dots, \chi^{r-1}$.

In [3] the idea of a sharp character was introduced (the character is sharp if $m = 1$ above) and a body of work has appeared related to sharp characters. The ideas have been generalised in several directions, including π-sharp characters where π is a set of primes ([5]), sharp classes ([14]), sharp triples ([8]), association schemes ([6]) and quasigroup characters ([4]). The *dimension* of a G-set X is defined in [13] to be the smallest k such that X^k contains a regular orbit. Motivated by (1) we have attempted here to connect the dimension of X to the set of values of the permutation character χ of X. Whereas the connection appears to be difficult to address for characters, if the set of marks associated to X replaces the character values we can provide an easy proof of an upper bound for the dimension of X using the Burnside ring of G.

The work led us to re-examine the original argument of Blichfeldt, and we give here a further generalisation for arbitrary pairs of characters. A remark of Strunkov, personal communication by Cameron on a result of Maillet and the work on the Burnside ring mentioned above suggest variations on the definition of a sharp character and other "minimal" objects. While experimenting with our original conjecture that the size of the value set of the permutation character is an upper bound on the dimension of a G-set X we were led to formulate a conjecture that certain polynomials in an arbitrary permutation character produce genuine characters, and whereas this again proved to be difficult to address we were able to prove the analogous conjecture in the context of the Burnside ring.

In Section 2 we introduce more formally the definition of a sharp character and indicate some of the results obtained, in particular those in the thesis of Maslyakov which appears not to be well known. We then give our generalisation of Blichfeldt's result which leads to a subtle restriction on group character tables. The Burnside ring and the table of marks of a group are introduced in Section 3; the proof is given of the result on the bound for the dimension of a G-set. In Section 4 we give the result on polynomials which produce G-sets, and in Section 5 we set out the various definitions of sharpness and give some of the relations between them.

2. Sharp characters

Let χ be a generalised character of a finite group G whose set of distinct values is $L = \{l_1, l_2, \dots, l_r\}$, $l_1 = \chi(e) = n$, with $V_i = \{g \in G : \chi(g) = l_i\}$ and $a_i = |V_i|$. It follows (see [2, 8, 14]) that for fixed i

$$a_i \prod_{j \neq i} (l_i - l_j) = b_\chi^i |G| \tag{2}$$

where b_χ^i is an algebraic integer which lies in $\mathbb{Z}$ if l_i is in $\mathbb{Z}$. In particular, if $\hat{L} = \{l_2, \dots, l_r\}$ is the set of values of χ on non-identity elements it follows that

$$f_{\hat{L}}(n) = m|G|, \qquad m \in \mathbb{Z}. \tag{3}$$

The generalised character χ is said to be *sharp* if $a_1 = 1$ and $m = 1$, and the triple $\{G, \chi, l_i\}$ is said to be *sharp* if b^i_χ is a unit in the ring of algebraic integers [8]. If χ is a faithful character it follows easily that $f_{\hat{L}}(\chi) = m\rho$, where ρ is the character of the regular representation, and the result of Burnside quoted above is a direct consequence. If χ is the character of a sharply k-transitive permutation representation then χ is sharp in this sense. In addition to the authors quoted above, Alvis in collaboration with others has produced work on sharp characters (see [1] and other references there). We also note the work from the thesis of Maslyakov which has appeared in [10] and [11] in which the following and other similar results are proved:

Theorem 2.1. *Let G be a p-group with a sharp irreducible character of degree p. Then $p = 2$ and G is dihedral or generalised quaternion.*

Maslyakov also makes the following interesting conjectures.

Conjecture 2.2. *If G has odd order with a sharp irreducible character then G is cyclic.*

Conjecture 2.3. *Let G be a simple group with a sharp irreducible character. Then G is one of the following: (a) cyclic, (b) alternating, (c) $PGL_2(2^n)$, $n \geq 2$, (d) $GL_2(2)$, (e) M_{11}, (f) M_{12}.*

As was mentioned in the Introduction, the result (3) has been extended by considering the values of a character on π-elements ([5]), and dualised to classes in [14]. The result (2) led to the definition of sharp triples in [8] and π-sharp characters are defined in [5]. Sharp characters of quasigroups are defined in [4] and of association scheme classes in [6]. It is pointed out in [4] that for all abelian groups and dihedral groups of twice odd prime order all irreducible triples are sharp.

It has been observed by Strunkov (personal communication) that if $\hat{L}_\pi$ is the set of values of χ on non-identity elements of prime power order then $f_{\hat{L}_\pi}(n) = m_\pi|G|$ where m_π is an integer, in general different from m. This follows by using the Blichfeldt result for the restriction of χ to the Sylow p-subgroup of G for each prime divisor p of $|G|$ which shows that for each prime p, $f_{\hat{L}_\pi}(n)$ is divisible by the highest power of p dividing $|G|$, and hence that $|G|$ divides $f_{\hat{L}_\pi}(n)$.

We give below another generalisation of Blichfeldt's original result.

Theorem 2.4. *Let χ be any generalised character of G, and let $L = \{l_1, \ldots, l_r\}$ be the set of distinct values of χ. Let $V_i = \{g \in G : \chi(g) = l_i\}$. Then if ψ is any character of G, and $\psi(V_i) = \sum_{g\in V_i} \psi(g)$, we have for arbitrary i*

$$\psi(V_i)\prod_{i\neq j}(l_i - l_j) = b_{l_i}|G|$$

where b_{l_i} is an algebraic integer.

Proof. Let χ be a generalised character of G and let ψ be a character of G. Let $\langle\chi, \psi\rangle_G$ denote the usual inner product of characters, i.e. $\langle\chi, \psi\rangle_G = |G|^{-1}\sum_{g\in G}\chi(g)\overline{\psi}(g)$. Then

$$|G|\langle 1, \overline{\psi}\rangle = \sum_{i=1}^{r} \psi(V_i)$$

$$|G|\langle \chi, \overline{\psi}\rangle_G = \sum_{i=1}^{r} l_i \psi(V_i)$$

$$|G|\langle \chi^2, \overline{\psi}\rangle_G = \sum_{i=1}^{r} l_i^2 \psi(V_i)$$

$$\ldots$$

$$|G|\langle \chi^{(r-1)}, \overline{\psi}\rangle_G = \sum_{i=1}^{r} l_i^{(r-1)} \psi(V_i).$$

The above equations form a linear system in the $\psi(V_i)$. Let u_i denote the integer $\langle \chi^i, \overline{\psi}\rangle_G$ and let $u_0 = \langle 1, \overline{\psi}\rangle$. On solving the system by Cramer's rule we obtain

$$\psi(V_i) = \frac{\Delta_i}{\Delta(l_1, \ldots, l_r)} \tag{4}$$

where $\Delta(l_1, \ldots, l_r)$ is the Vandermonde determinant of $l_1, \ldots, l_r$, and Δ_i is obtained from $\Delta(l_1, \ldots, l_r)$ by replacing the ith column with the column

$$(|G|u_o, \ldots, |G|u_{r-1}).$$

The Laplace expansion of Δ_i with respect to the ith column leads to

$$\Delta_i = |G| \sum_{m=0}^{r-1} (-1)^{m+1+i} u_m \Delta_{im} \tag{5}$$

where Δ_{im} is the appropriate cofactor. We let $\alpha_r(x_1, x_2, \ldots, x_t)$ denote the rth elementary symmetric function on $\{x_i\}$ with the convention that $\alpha_0 = 1$. Now it is well known that

$$\Delta_{im} = \alpha_{r-1-m}(l_1, \ldots, l_{i-1}, l_{i+1}, \ldots l_r)\Delta(l_1, \ldots, l_{i-1}, l_{i+1}, \ldots l_r).$$

Direct substitution into (5) and rearrangement of (4) implies that

$$\psi(V_i)\frac{\Delta(l_1, \ldots, l_r)}{\Delta(l_1, \ldots, l_{i-1}, l_{i+1}, \ldots l_r)}$$
$$= |G| \sum_{m=0}^{r-1} (-1)^{m+1+i} \alpha_{r-1-m}(l_1, \ldots, l_{i-1}, l_{i+1} \ldots l_r) u_m$$

which leads to the result of the theorem. □

Note that if $l_i \in \mathbb{Z}$ the above proof implies that b_{l_i} is an integer. In particular we have the following corollary.

Corollary 2.5. *With the notation of the theorem, let* $l_1 = 0$. *Then*

$$\psi(V_1) \prod_{i=2}^{r} l_i = b_0 |G|$$

where b_0 *is an integer.*

We remark that there are dual results to Theorem 2.4 and Corollary 2.5 which are valid in the context of association schemes. We leave the formulation to the reader (see [6]).

The result above gives the following restriction on characters which may be worthy of note. If the character χ takes the value 0 on the subset U of G, the product of the non-zero values of χ is s and if ψ is any other character of G then the sum of the values of ψ on U is either 0 or of absolute value at least $|G|/s$.

3. The Burnside ring and dimensions of *G*-sets

We refer to [7] for a full discussion of the Burnside ring. Let G be a group. As is well known, the set $\mathcal{D}$ of G-isomorphism classes of transitive G-sets $X_1, \dots, X_t$ is in 1:1 correspondence with the set of conjugacy classes $\{\mathcal{U}_i\}$ of subgroups of G, X_i being associated with $U_i = \{g \in G : g(x) = x\}$ for some fixed $x \in X_i$. The *Burnside ring* Ω_G is the Grothendieck ring of $\mathcal{D}$. It consists of formal linear combinations

$$\sum n_i X_i$$

with $n_i \in \mathbb{Z}$, the operation of addition being defined by the disjoint union

$$X_i + X_j = X_i \cup X_j$$

and that of multiplication being the direct product

$$X_i \cdot X_j = X_i \times X_j$$

with G acting by $g(x_1, x_2) = (g(x_1), g(x_2))$, both operations being extended linearly to Ω_G. The identity element of Ω_G which corresponds to $U_i = \{e\}$ will be denoted by 1.

Consider a G-set X. If U is any subgroup of G let X_U be defined by

$$X_U = \{x \in X \mid \forall g \in U, g(x) = x\}.$$

Then $|X_U|$ is the *mark* of U on X, and it is clear that any subgroup conjugate to U has the same mark on X. The information was encoded by Burnside into the table of marks. A set of representatives $\{U_i\}$ is chosen where $U_i \in \mathcal{U}_i$ and is ordered such that $|U_i| \le |U_j|$ for $i < j$. The *table of marks* is the square array indexed by the set $\{U_i\}$ whose (i, j)th entry is $m_i(X_j)$, the mark of U_i on X_j. For example, the table of marks for the symmetric group S_3 is

$$\begin{array}{cccc} 6 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1. \end{array}$$

Thus to each G-set X there is associated a vector of marks whose ith entry is the mark of U_i on X (if X is a transitive G-set this is the corresponding column in the table of marks). The marks naturally extend to an element $Y = \sum_{j=1}^{t} c_j X_j$ of Ω_G, the ith component of the mark vector being $\sum_{i=1}^{t} c_j m_i(X_j)$. It is clear that all the entries of the mark vector of a G-set are non-negative, but a non-negative mark vector does not ensure that an element of Ω_G is a genuine G-set. In the following we will use the fact that the table of marks is upper triangular with non-zero entries in the diagonal. This implies in particular that if Y is a G-set with X_i as a constituent then the mark of U_i on Y is non-zero. The G-sets X and Y are G-isomorphic if and only if their mark vectors are equal. We note that there is a homomorphism $\Omega_G \to R(G)$ where $R(G)$ is the character ring of G, but that this homomorphism has a kernel for non-cyclic groups. It may be verified that the diagonal entry in the ith place of the table of marks is $|N_G(U_i)|/|U_i|$. We remark also that if χ is the character of the G-set X ($\chi(g) = |\text{fix}(g)|$) then the set of values of χ is the set of marks of cyclic subgroups on X. A G-set will be said to be *faithful* if $g(x) = x$ for each x in X implies $g = e$.

We remind the reader that the dimension of G-set X the least integer k such that X^k contains a regular orbit (where the multiplication is in Ω_G). It is also the size of the smallest base in the sense of computational group theory (see for example [12]). The following theorem is equivalent to a result in [13], but our proof is very direct and in addition gives an explicit representation of the regular G-set as a polynomial in X.

Theorem 3.1. *If X is a faithful G-set with $\hat{M} = \{m_2, \ldots, m_r\}$ the set of distinct marks of non-identity subgroups on X then the dimension of X is at most $r - 1$.*

Proof. Suppose $n = |X|$. Consider the generalised G-set

$$f_{\hat{M}}(X) = (X - m_2)(X - m_3) \cdots (X - m_r).$$

The marks of all non-identity subgroups are zero, and the mark of $\{e\}$ is

$$f_{\hat{M}}(n) = (n - m_2)(n - m_3) \cdots (n - m_r).$$

But if χ is the character of X, and $\hat{L} = \{l_2, \ldots, l_s\}$ is the set of distinct values of χ on non-identity elements it follows from the Blichfeldt result that $f_{\hat{L}}(n) = b_{\hat{L}}|G|$ with $b_{\hat{L}} \in \mathbb{Z}$, and since $\hat{L} \subset \hat{M}$ it follows that the mark vector of $f_{\hat{M}}(X)$ is the same as that of bX_ρ, where X_ρ is the regular G-set and b is a positive integer, i.e. $f_{\hat{M}}(X) = bX_\rho$ so $f_{\hat{M}}(X)$ is a genuine G-set. Now since $f_{\hat{M}}(X) = \sum_{i=0}^{r} a_i X^i$ with $a_i \in \mathbb{Z}$ it follows that X_ρ is contained in X^i for some $i \leq r$. $\square$

Since the character of a permutation is in general much easier to obtain than the set of marks, it would be convenient if the following conjecture were true.

Conjecture 3.2. *Let X be a faithful G-set with $L = \{l_1, \ldots, l_s\}$ the distinct set of values of the character χ of X. Then the dimension of X is at most s.*

4. *G*-sets as polynomials

Let X be a G-set with corresponding permutation character χ. If $f(x)$ is any polynomial with integer coefficients it is clear that $f(X)$ is an element of Ω_G. However, it appears to be difficult to determine whether $f(X)$ is a genuine G-set. In the case where the set of distinct values of χ is $N_r = \{0, 1, 2, \ldots, r\}$ a G-set can be constructed as follows: suppose $s \leq r + 1$ and let G act diagonally on the set Y_s of s-tuples of distinct elements of X, i.e. $Y_s = \{(x_{i_1}, x_{i_2}, \ldots, x_{i_s}), x_{i_j} \neq x_{i_k}$ for $j \neq k\}$ with $g(x_{i_1}, x_{i_2}, \ldots, x_{i_s}) = (gx_{i_1}, gx_{i_2}, \ldots, gx_{i_s})$ for $g \in G$. Then it is easy to check via the mark vector that

$$Y_s = X(X-1)(X-2)\cdots(X-s+1)$$

and thus $f_{N_{s-1}}(X)$ is a genuine G-set. The following is a generalisation.

Theorem 4.1. *Let X be a faithful G-set with $M^* = \{m_1, \ldots, m_r\}$ the set of distinct marks of non-identity subgroups on X where $m_i < m_j$ for $i < j$ (and thus $m_1 = 0$). Then if $s \leq r+1$, $f_s(X) = X(X-m_2)(X-m_3)\cdots(X-m_{s-1})$ is a genuine G-set.*

Proof. Let $Y_t = f_t(X) = X(X-m_2)(X-m_3)\cdots(X-m_{t-1})$. Assume Y_t is a genuine G-set for some $t < r+1$. Let Z be an irreducible G-set which is a constituent of Y_t. Consider XZ. Suppose the mark vector of Z is

$$[b_1, b_2, \ldots, b_j, 0, \ldots 0]^{\mathrm{T}}$$

and that of X is

$$[a_1, a_2, \ldots, a_i, 0, \ldots 0]^{\mathrm{T}}$$

where of course each $a_k = m_l$ for some l. Since Z must be a constituent of X^s for some $s \leq t$ we must have $j \leq i$. Then the mark vector of XZ is

$$[a_1b_1, a_2b_2, \ldots, a_jb_j, 0, \ldots 0]^{\mathrm{T}}.$$

Now since this is a genuine G-set its decomposition into irreducible G-sets must be

$$XZ = a_jZ + V$$

where V is a genuine G-set. We claim that $a_j \geq m_t$, for if $a_j < m_t$ the mark of the subgroup U_j on Y_t which is $a_j(a_j-m_2)(a_j-m_3)\cdots(a_j-m_{t-1})$ is 0 contradicting that Z is a constituent of the genuine G-set Y_t. It follows that $XZ = m_tZ + V'$ where V' is a genuine G-set and hence that

$$XZ - m_tZ = (X-m_t)Z$$

is a genuine G-set. Thus $Y_t(X-m_t) = Y_{t+1}$ is a genuine G-set, and the proof of the theorem follows by induction (the first step being the trivial statement that X is a genuine G-set). □

It would appear to be interesting to give a direct combinatorial description of the sets Y_t.

Conjecture 4.2. *Let X be a faithful transitive G-set with character χ and $L^* = \{l_1, \ldots, l_k\}$ be the distinct set of values of χ on non-identity elements where $l_i < l_j$ for $i < j$. Then if $s \leq k+1$, $f_s(\chi) = \chi(\chi-l_2)(\chi-l_3)\cdots(\chi-l_{s-1})$ is a genuine character of G.*

We note that the conjecture is equivalent to Theorem 4.1 if $M^* = L^*$. If the mark $m_i \notin L^*$ it is clear that $f_s(X)$ has negative marks for $s \geq i$ and cannot be a genuine G-set.

5. Some comments on sharpness

First, we assume that χ is an arbitrary generalised character of G. In the light of the comment of Strunkov mentioned above we make the following definition. Suppose $\hat{L}_\pi$ is the set of values of χ on non-identity elements of prime power order.

Definition 5.1. The character χ is S-sharp if $f_{L_\pi}(n) = |G|$.

Although as was mentioned above the integer m_π is in general less than m we do not have examples of S-sharp generalised characters which are not sharp. If we now assume that χ is a permutation character corresponding to the G-set X there are other ways in which the definition of sharpness can be modified. The first discussion of Blichfeldt-type results occurred as early as 1895 in [9] (this reference was brought to our attention by Cameron). There Maillet showed that if X is a faithful G-set and $\hat{M}_\pi$ is the set of distinct marks of non-identity subgroups of prime power order on X then $f_{\hat{M}_\pi}(n) = b_{\hat{M}_\pi}|G|$ with $b_{\hat{M}_\pi} \in \mathbb{Z}$. Since Maillet's proof is quite involved we sketch a quick proof using the ideas here. If P is an arbitrary Sylow p-subgroup of G of order $p^{\alpha(p)}$ then X is a faithful P-set and hence by the argument in Theorem 3.1 if $\hat{M}_p$ is the set of distinct marks of non-identity subgroups of P on X then

$$f_{\hat{M}_p}(X) = b_p X_\rho^p$$

where X_ρ^p is the regular representation of P and $b_p \in \mathbb{Z}$ and hence

$$f_{\hat{M}_p}(n) = b_p|P|.$$

It follows that $f_{\hat{M}_\pi}(n)$ is an integer divisible by $p^{\alpha(p)}$ for all p dividing $|G|$ which implies Maillet's result. We can thus define a third version of "sharpness".

Definition 5.2. The G-set X is M-sharp if $f_{\hat{M}_\pi}(n) = |G|$.

Given the result in Theorem 3.1 we can make a fourth definition. As above, let $\hat{M}$ be the set of distinct marks of non-identity subgroups of G on X.

Definition 5.3. The G-set X is B-sharp if $f_{\hat{M}}(n) = |G|$.

It is clear that if X is a faithful G-set with character χ then the following hold: X B-sharp implies X M-sharp, χ S-sharp and χ sharp. X M-sharp implies χ S-sharp. χ sharp implies χ S-sharp.

We recall the following definition of Matsuhisa. Let χ be a character of a finite group G whose set of distinct values is $L = \{l_1, l_2, \ldots, l_r\}$, $l_1 = \chi(e)$, with $a_i = |\{g : \chi(g) = l_i\}|$.

Definition 5.4. (G, χ, l_i) is a sharp triple if $a_i \prod_{j \neq i}(l_i - l_j) = u|G|$ where u is a unit.

In the light of the result in Theorem 2.4, we make the following definitions which generalise that above. Let L be as above the set of distinct values of the generalised character χ.

Definition 5.5. Let ψ be a character of G. (G, χ, l_i, ψ) is a sharp quadruple if $\psi(V_i)\prod_{j\neq i}(l_i - l_j) = u|G|$ where u is a unit.

Definition 5.6. (G, χ, ψ) is a 0-sharp triple if $\psi(V_1)\prod_{j>1} l_j = u|G|$ where u is a unit.

Example 5.7. Let $G = PSL(2, 7)$. Its character table is

Class $\lvert Cl(g)\rvert$	C_1 1	C_2 21	C_4 42	C_3 56	C_{7_1} 24	C_{7_2} 24
χ_1	1	1	1	1	1	1
χ_2	6	2	0	0	-1	-1
χ_3	7	-1	-1	1	0	0
χ_4	8	0	0	-1	1	1
χ_5	3	-1	1	0	α	$\bar{\alpha}$
χ_6	3	-1	1	0	$\bar{\alpha}$	α

where $\alpha = (-1+\sqrt{7}\mathrm{i})/2$. It is easily calculated that no triple of the form $(G, \chi_i, 0)$ where χ_i is irreducible is sharp. However, all the triples (G, χ_2, χ_3), (G, χ_3, χ_5) and (G, χ_4, χ_5) are 0-sharp.

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